Probability Theory

12/01/22

$i = [0 \dots N-1]$

T_{ij} prob. of going from j to i in one step.

$p_0(i)$ prob of being in i at time 0.

$$p_1(i) = \sum_j T_{ij}\, p_0(j)$$

$$\vec{p}_{t+1} = T\, \vec{p}_t$$

$$T_{ij} \geq 0 \qquad \sum_i T_{ij} = 1 \quad \forall j$$

$$p_0(i) \geq 0 \qquad \sum_i p_0(i) = 1$$

$$\Downarrow$$

$$p_t(i) \geq 0 \qquad \sum_i p_t(i) = 1$$

$T \geq 0$ T positive.

T always has a eigenvalue 1

$\exists\ \vec{p}$ such that

$$T\vec{p} = \vec{p}$$

$$\vec{p}_0 \neq \vec{p}$$

$$T^n \vec{p}_0 \to \vec{p}$$

$\vec{p}_1 \quad \vec{p}_2$
$$\|\vec{p}_1 - \vec{p}_2\| = \sum_i |p_1(i) - p_2(i)|$$

For every $\vec{p}_0$

$$\|T^n \vec{p}_0 - \vec{p}\| \leq C e^{-\alpha n}$$

if $T > 0$, $T_{ij} > 0 \quad \forall i,j$.

Random Walk

$$T_{ii} = q \qquad T_{i,i+1} = T_{i,i-1} = \frac{p}{2}$$

R_{ij} The Transition matrix for The random walk

$$\left(R^N\right)_{ij} > 0 \qquad \mathring{\vec{p}} = \left(\frac{1}{N} \; \dots \; \frac{1}{N}\right)$$

$$T = R^N$$

$$\| T^n \vec{p}_0 - \mathring{\vec{p}} \| \leq C e^{-\kappa n}$$

$$\| R^{Nn} \vec{p}_0 - \mathring{\vec{p}} \| \leq C e^{-\kappa n}$$

$$\Downarrow$$

$$\| R^n \vec{p}_0 - \mathring{\vec{p}} \| \leq C e^{-\frac{\kappa}{N} n}$$

if There exists N such That $(T^N)_{ij} > 0 \;\; \forall_{ij}$

[~~for every~~ i,j $\exists N$ such $(T^N)_{ij} > 0$]

$$T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$T^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$T = S + \lambda \bar{T} \qquad \bar{T} = \vec{p} \otimes \mathbb{1}$$

$$\bar{T}_{ij} = \vec{p}_i$$

$$T\vec{p} = \vec{p}$$

$$S\vec{p} = T\vec{p} - \lambda \bar{T}\vec{p} = (1-\lambda)\vec{p}$$

$$\bar{T}\vec{p} = \vec{p}$$

$$T\vec{p}_0 = S\vec{p}_0 + \lambda \bar{T}\vec{p}_0 = S\vec{p}_0 + \lambda \vec{p}$$

$$T^2 p_0 = T\left(S\vec{p}_0 + \lambda \vec{p}\right) =$$

$$= S^2\vec{p}_0 + \lambda \bar{T} S p_0 + \lambda T\vec{p} =$$

$$= S^2\vec{p}_0 + \lambda S\bar{T}\vec{p}_0 + \lambda \vec{p} =$$

$$= S^2 \vec{p}_0 + \lambda S \vec{p} + \lambda \vec{p} =$$

$$= S^2 \vec{p}_0 + \lambda (1-\lambda) \vec{p} + \lambda \vec{p}$$

$$T^n \vec{p}_0 = S^n \vec{p}_0 + \lambda \sum_{i=0}^{n-1} (1-\lambda)^i \vec{p} =$$

$$= S^n \vec{p}_0 + (1 - (1-\lambda)^n) \vec{p}$$

Assume that $S_{ij} \geq 0$

$$T - \lambda \bar{T} \geq 0$$

$$\sum_i S_{ij} = \sum_i T_{ij} - \lambda \bar{T}_{ij} = (1-\lambda)$$

$$\| S p \| = \sum_i \Big| \sum_j S_{ij} \, p(j) \Big| \leq$$

$$\leq \sum_i \sum_j S_{ij} |p(j)| =$$

$$= \sum_j \Big(\sum_i S_{ij} \Big) |p(j)| =$$

$$= (1-\lambda) \| p \|$$

$$\| T^n \vec{p}_0 - \vec{p} \| \leq \| S^n p_0 \| + (1-\lambda)^n \vec{p} \leq$$

$$\leq 2 (1-\lambda)^n = C e^{-\varkappa n}$$

$$\varkappa = -\log (1-\lambda)$$

$$T - \lambda \vec{p}$$

$$\lambda = \min_{i,j} T_{ij} \Big/ \max_i p(i)$$

$$p_{t+dt}(i) = (1-p)\, p_t(i) + \frac{p}{2} p_t(i-1) + \frac{p}{2} p_t(i+1)$$

$$p_{t+dt}(i) - p_t(i) = \frac{p}{2} \left(p_t(i-1) - 2p(i) + p_t(i+1) \right)$$

$$x = i\,dx$$

$$= \frac{p}{2} \left(p(x-dx) - 2p(x) + p(x+dx) \right)$$

$$\frac{p(x+dx) - 2p(x) + p(x-dx)}{dx^2} \to p''(x)$$

$$\frac{p(x+dx)-p(x)}{dx} \simeq p'\left(x+\frac{dx}{2}\right)$$

$$\frac{p(x-dx)-p(x)}{dx} \simeq -p'\left(x-\frac{dx}{2}\right)$$

$$\frac{p'\left(x+\frac{dx}{2}\right)-p'\left(x-\frac{dx}{2}\right)}{dx} = p''(x)$$

$$dt = dx^2$$

$$\mu = \frac{p}{2}$$

$$\frac{d}{dt} p_t(x) = \mu \; p_t''(x)$$

Heat Eq.

Diff. Eq.

$$X_\tau = \begin{cases} +1 & \frac{p}{2} \\ 0 & 1-p \\ -1 & \frac{p}{2} \end{cases}$$

$$p = \sum_{i=0}^{T} X_\tau \simeq N(0, \sigma^2)$$

Sol. heat eq.

$$p_t(x) = \int \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x-y}{2\sigma^2}} p_0(y)\,dy$$

$\sigma^2 = ct$